MATH 2050 - Monotone Convergence Theorem

(Reference : Bartle § 3.3)

Q: Can we determine the limit of (x_n) exist <u>without</u> knowing the value of the limit ?

<u>Recall</u>: (x_n) convergent $\underset{\not\Leftarrow}{\Rightarrow}$ (x_n) bdd

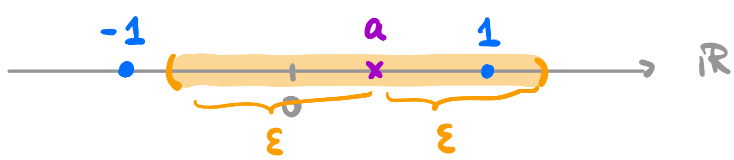
false

$\Rightarrow$ <u>Cor</u>: (x_n) unbdd $\Rightarrow$ (x_n) divergent. "Divergence Test"

<u>Couterexample</u>: $(x_n) = ((-1)^n)$ is bdd BUT divergent

<u>Pf</u>: Suppose <u>(x_n) is convergent</u>, say $\lim(x_n) = a \in \mathbb{R}$.

Take $\varepsilon = 1$, $\exists\, K \in \mathbb{N}$ s.t. $|x_n - a| < \varepsilon = 1 \quad \forall\, n \geq K$



For $n \geq K$ is odd, we have

$$|x_n - a| = |-1 - a| < 1$$

$$\Rightarrow \quad -2 < a < 0$$

For $n \geq K$ is even, we have

$$|x_n - a| = |1 - a| < 1$$

$$\Rightarrow \quad 0 < a < 2$$

Contradiction!

$\blacksquare$

Q: Under what condition(s) does

$$(x_n) \text{ bdd} \Rightarrow (x_n) \text{ convergent ?}$$

Monotone Convergence Theorem (MCT)

$$(x_n) \text{ bdd} + \text{monotone} \Rightarrow (x_n) \text{ convergent}$$

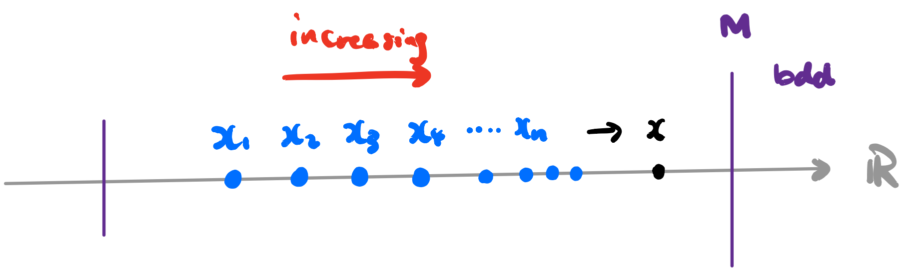
Def$^{\underline{n}}$: (x_n) is monotone if it is

either (i) increasing, i.e. $x_1 \le x_2 \le x_3 \le \cdots$ $(x_n \le x_{n+1} \ \forall n \in \mathbb{N})$

or (ii) decreasing, i.e. $x_1 \ge x_2 \ge x_3 \ge \cdots$ $(x_n \ge x_{n+1} \ \forall n \in \mathbb{N})$

Note: If inequalities are strict, then we say it is strictly monotone / increasing / decreasing.

Picture:



Proof of MCT: Idea: $\lim (x_n) = \sup \{x_n \mid n \in \mathbb{N}\}$

Suppose (x_n) is bdd and increasing. Consider

$$\phi \neq S := \{x_n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$$

Note (x_n) is bdd $\Rightarrow$ S is bdd above & below

By completeness of $\mathbb{R}$, $x := \sup S$ exists.

Claim: $\lim (x_n) = x$

Pf of Claim: We show this using ε- K def^{n} of limit.

Let $\varepsilon > 0$ be fixed but arbitrary.

Since $x = \sup S$, $x - \varepsilon$ CANNOT be an upper bd for S

ie $\exists K \in \mathbb{N}$ s.t. $x - \varepsilon < x_K$

Since (x_n) is increasing (i.e. $x_n \leq x_{n+1}$ $\forall n \in \mathbb{N}$)

$\Rightarrow$ ①: $x - \varepsilon < x_K \leq x_{K+1} \leq x_{K+2} \leq \cdots \leq x_n \quad \forall n \geq K$

On the other hand, $x = \sup S$ is an upper bd for S

$\Rightarrow$ ②: $x_n \leq x < x + \varepsilon \quad \forall n \in \mathbb{N}$

Combining ① & ②.

$$x - \varepsilon < x_n < x + \varepsilon \quad \forall n \geq K$$

$\square$

Example 1 "Harmonic series"

Let $h_n := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$, $n \in \mathbb{N}$.

ie $h_1 = 1$, $h_2 = 1 + \frac{1}{2} = \frac{3}{2}$, ...

Show that (h_n) is divergent.

Pf: Note $h_{n+1} = h_n + \frac{1}{n+1} > h_n \quad \forall n \in \mathbb{N}$

ie (h_n) is strictly increasing!

By MCT, (h_n) divergent $\Longleftrightarrow$ (h_n) unbdd

Claim: (h_n) is unbdd!

Consider $n = 2^m$, $m \in \mathbb{N}$.

$$h_{2^m} = \underbrace{1}_{1\text{ term}} + \underbrace{\tfrac{1}{2}}_{1\text{ term}} + \underbrace{\left(\tfrac{1}{3} + \tfrac{1}{4}\right)}_{2\text{ term}} + \underbrace{\cdots}_{4\text{ terms}} + \underbrace{\left(\tfrac{1}{2^{m-1}+1} + \cdots + \tfrac{1}{2^m}\right)}_{2^{m-1}\text{ terms}}$$

$$> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \cdots + \left(\tfrac{1}{2^m} + \cdots + \tfrac{1}{2^m}\right)$$

$$= 1 + \underbrace{\tfrac{1}{2} + \tfrac{1}{2} + \cdots + \tfrac{1}{2}}_{m\text{ terms}} = 1 + \frac{m}{2}$$

unbdd as $m \to \infty$.

$h_1 = 1$

$h_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}}$

$\Rightarrow$ (h_n) is unbdd.

Remark: MCT works well for recursive sequence.

Example 2: Let (y_n) be defined "recursively" by:

$$y_1 := 1 \; ; \quad y_{n+1} := \frac{1}{4}(2y_n + 3) \quad (*) \quad \forall n \in \mathbb{N}$$

Show that $\lim(y_n) = \frac{3}{2}$.

Proof: General Strategy

- Step 1: Apply MCT to show the limit first
- Step 2: Take limit in the recursive relation $(*)$ to compute the limit of the seq.

We first show that (y_n) is bdd & monotone.

Claim: (y_n) is bdd above by 2.

Pf of Claim: Use M.I. Note $y_1 := 1 < 2$.

Suppose $y_k < 2$. Then, $y_{k+1} = \frac{1}{4}(2y_k + 3) < \frac{7}{4} < 2$.

$y_1 = 1$

$y_2 = \frac{1}{4}(2+3) = \frac{5}{4}$

$y_3 = \frac{1}{4}\left(2 \cdot \frac{5}{4} + 3\right) = \frac{11}{8}$

Claim: (y_n) is increasing, i.e. $y_n \le y_{n+1}$ $\forall n \in \mathbb{N}$.

Pf of Claim: Use M.I. Note $y_1 := 1 < \frac{5}{4} = y_2$.

Assume $y_k \le y_{k+1}$. Then

$$y_{k+1} = \frac{1}{4}(2y_k + 3) \le \frac{1}{4}(2y_{k+1} + 3) = y_{k+2}.$$

So (y_n) is bdd & monotone. by MCT, $\lim (y_n) = y$ exists.

Since (y_n) is convergent, we have $\lim (y_{n+1}) = \lim (y_n) = y$

Take $n \to \infty$ on both sides of (*):

$$\lim (y_{n+1}) = \lim \frac{1}{4}(2y_n + 3) \underset{\substack{\uparrow \\ \text{Limit} \\ \text{Thm}}}{=} \frac{1}{4}(2 \lim (y_n) + 3)$$

$$\Rightarrow \quad y = \frac{1}{4}(2y + 3)$$

Solving for y, get $y = \frac{3}{2}$.

Example 3: Fix $a > 0$. Define inductively

$$S_1 := 1 \quad ; \quad S_{n+1} := \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) \quad (**) \quad \forall n \in \mathbb{N}$$

Show that $\lim (S_n) = \sqrt{a} > 0$.

Proof: Claim 1: (S_n) is bdd below by $\sqrt{a}$ (for $n \ge 2$)

Pf of Claim: Note $S_n > 0$ $\forall n \in \mathbb{N}$. Rewrite (**) as

$$S_n^2 - 2S_{n+1}S_n + a = 0$$

So, $x^2 - 2S_{n+1}x + a = 0$ has at least a real root S_n

$$\Rightarrow \quad 4S_{n+1}^2 - 4a \ge 0 \quad \Rightarrow \quad S_{n+1} \ge \sqrt{a} \quad \forall n \in \mathbb{N}$$

Claim 2: (s_n) is decreasing "eventually", ie $s_n \geq s_{n+1} \quad \forall n \geq 2$.

Pf of Claim: $\forall n \geq 2$,

use Claim 1

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) = \frac{1}{2}\left(\frac{s_n^2 - a}{s_n}\right) \geq 0$$

By MCT, $\lim(s_n) =: s$ exists.

Take $n \to \infty$ on both sides of $(**)$, then we obtain

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right)$$

$\left(\begin{array}{l} \underline{\text{Note}}: \; s_n \geq \sqrt{a} \quad \forall n \geq 2 \\ \Rightarrow \; s \geq \sqrt{a} > 0 . \end{array}\right)$

Solve for s

$\Rightarrow \quad s = \sqrt{a} > 0$.

∎

Def^n: Let $(x_n)_{n \in \mathbb{N}}$ be a seq. of real numbers.

Suppose $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing seq. of natural no..

THEN,

$$(x_{n_k})_{k \in \mathbb{N}} := (x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of $(x_n)_{n \in \mathbb{N}}$.

k^{th} term of (x_{n_k}) $=$ n_k^{th} term of (x_n)

Intuitively:

$$(x_n) = (x_1, x_2, x_3, x_4, x_5, x_6, \ldots)$$

$$(x_{n_k}) = (x_1, x_2, x_4, x_6, \ldots \ldots)$$

$k=1$	$k=2$	$k=3$	$k=4$
$n_1=1$	$n_2=2$	$n_3=4$	$n_4=6$

E.g.) (Tail of a seq.) For each fixed $\ell \in \mathbb{N}$, then

the ℓ-tail $(x_{k+\ell})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$

(Here, $n_k = k + \ell$)

E.g.) $(x_n) = ((-1)^n)$

Then $(1, 1, 1, \dots, 1, \dots)$ is a subseq.